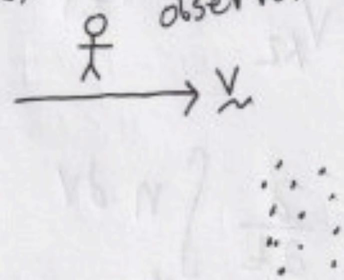
MIT OCW GR PSET 2

1. · Show that the number density of dust measured by an observer whose 4-velocity is $\vec{U}$ is given by $n = -\vec{N}\cdot\vec{U}$, where $\vec{N}$ is the matter current 4-vector:

in observer's IRF

observer flies by



Dust Stationary

$$\vec{N} \equiv (n, n\underset{\sim}{v})$$

$$\vec{U} = \text{[redacted]}\ (1, \underset{\sim}{0})$$

$$\vec{N}\cdot\vec{U} = \text{[redacted]}$$

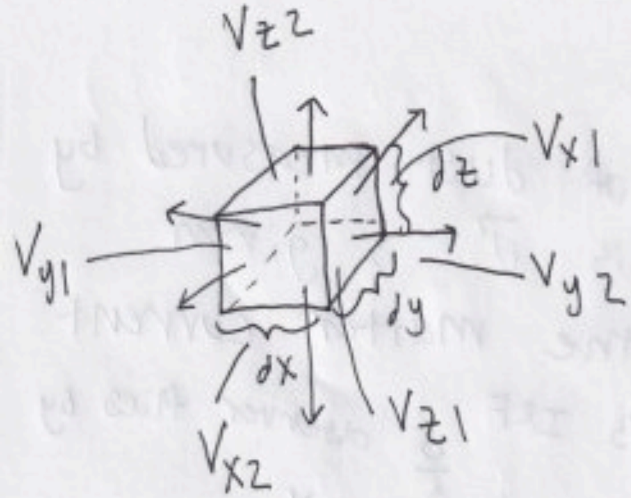
$$-n\cdot 1 + n\underset{\sim}{v}\cdot\underset{\sim}{0} = -n$$

$$\Rightarrow \boxed{n = -\vec{N}\cdot\vec{U} \text{ for all observers}} \quad ✓$$

2. · Take the limit of the continuity equation for $|\underset{\sim}{v}| \ll 1$ to get that $\frac{\partial n}{\partial t} + \frac{\partial (n v^i)}{\partial x^i} = 0$

· i.e. this is just asking to convert integral form to derivative form:

$$\frac{\partial}{\partial t}\int_{V^3} n\, dV = \int_{\partial V^3} n\underset{\sim}{v}\cdot d\underset{\sim}{a}$$



busy picture but consider infinitessimal cube with area vectors $d\underset{\sim}{a}_i$ on each side + the vector field changing slowly in this limit s.t. $dV_x = dV_{x2} - dV_{x1}$ etc

$$\Rightarrow \frac{\partial}{\partial t} \int_{V_3} n \, dV = - \oint_S n \underset{\sim}{v} \cdot d\underset{\sim}{a}$$

$$\rightarrow \frac{\partial n}{\partial t} \underbrace{\left(\int_{V_3} dV \right)}_{= dx\,dy\,dz} = \text{[redacted]} - n\underset{\sim}{v} \cdot \oint_S d\underset{\sim}{a}$$

minus sign since $d\underset{\sim}{a}_i$ points on this side in opposite [redacted] direction

$$\rightarrow dx\,dy\,dz \frac{\partial n}{\partial t} = -n\Big((V_{x2} - V_{x1})\hat{x} + (V_{y2} - V_{y1})\hat{y} + (V_{z2} - V_{z1})\hat{z} \Big) \cdot \left(dy\,dz\,\hat{x} + dx\,dz\,\hat{y} + dx\,dy\,\hat{z} \right)$$

$$\rightarrow dx\,dy\,dz \frac{\partial n}{\partial t} = -n \left(dV_x \, dy\,dz + dV_y \, dx\,dz + dV_z \, dx\,dy \right)$$

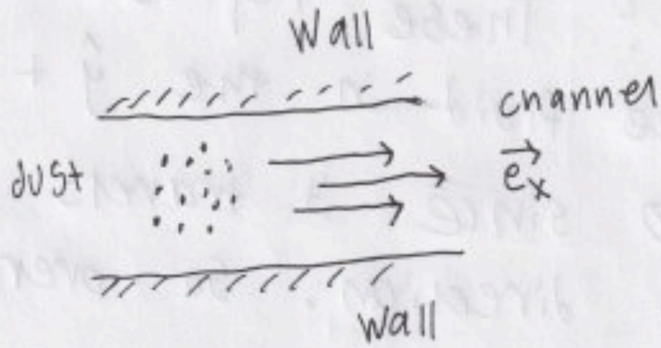
$$\rightarrow \frac{\partial n}{\partial t} = -n \left(\frac{dV_x}{dx} + \frac{dV_y}{dy} + \frac{dV_z}{dz} \right)$$

$$\rightarrow \boxed{\frac{\partial n}{\partial t} + \frac{\partial (n v^i)}{\partial x^i} = 0}$$

✓

$\boxed{3}$. In an inertial frame $\mathcal{O}$, calculate the components of the stress-energy tensor of the following systems:

$\boxed{a}$. A group of particles all moving with the same 3-velocity $\vec{V} = \beta \vec{e}_x$ as seen in $\mathcal{O}$. Let the rest-mass density of these particles be ρ_0 as measured in their own rest frame.



· ρ_0 gets Lorentz boosted since their is both spatial contraction + $m_0 \to \gamma m_0$

$$\Rightarrow \quad \rho_0 \to \gamma^2 \rho_0$$

· See the lectures notes to find that

$$T^{00} = \gamma^2 \rho_0, \quad T^{0i} = \gamma^2 \rho_0 V^i = T^{i0}, \text{ and}$$

$$T^{ij} = \gamma^2 \rho_0 V^i V^j$$

$$\vec{V} = (V^i, V^j, V^k) = (\beta, 0, 0)$$

· So we can start to fill in some parts of the matrix:

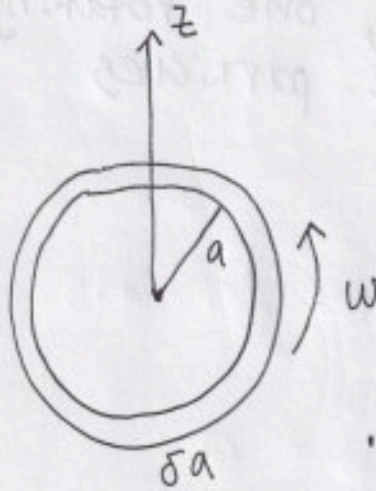
$$\overline{\overline{T}} = \gamma^2 \rho_0 \begin{bmatrix} 1 & \beta & ? & ? \\ \beta & \beta^2 & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

· The circled parts equal $\emptyset$ since $v^j = 0$ for all these terms...

· But what about the last two diagonal elements T_{yy} + T_{zz}? These represent pressure exerted by the fluid in the $\hat{y}$ + $\hat{z}$ directions which is $\emptyset$ since it travels uniformly in the $\hat{x}$ direction. So overall

$$\boxed{\overline{\overline{T}} = \gamma^2 \rho_0 \begin{bmatrix} 1 & \beta & 0 & 0 \\ \beta & \beta^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

b · A ring of N particles of rest mass "m" rotating counter-clockwise in the x-y plane about the origin of $\mathcal{O}$ at a radius "a" from this point with angular velocity ω. The ring is a torus of circular cross-section $\delta a \ll a$. Part of this calculation should relate ρ_0 in terms of the known quantities:



- The 4-velocity for a given dust grain is:

ccw motion

$$\vec{u} = \gamma(1, -\omega y, \omega x, 0)$$

- We know that for [redacted] dust the stress energy tensor is:

$$T^{\alpha\beta} = \rho_0 U^\alpha U^\beta$$

$$= \boxed{\rho_0 \gamma^2 \begin{bmatrix} 1 & -\omega y & \omega x & 0 \\ -\omega y & \omega^2 y^2 & -\omega^2 xy & 0 \\ \omega x & -\omega^2 xy & \omega^2 x^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = T^{\alpha\beta}}$$

$$\rho_0 = \frac{\text{rest mass}}{\text{proper volume}} = \boxed{\frac{Nm}{\gamma(2\pi a)\pi\,\delta a^2} = \rho_0}$$

factor of gamma here since observer views the ring contracted tangential to rotation direction so the "proper" ρ_0 is γ "smaller" than just $Nm/2\pi^2 a\,\delta a^2$

c. Two such rings of particles, one rotating clockwise + the other CCW. The particles do not collide or interact...

· for a CW rotating dust torus:

$$\vec{u} = \gamma(1, \omega y, -\omega x, 0)$$

· Use $T^{\alpha\beta} = \rho_0 u^\alpha u^\beta$

$$= \rho_0 \gamma^2 \begin{bmatrix} 1 & \omega y & -\omega x & 0 \\ \omega y & \omega^2 y^2 & -\omega^2 xy & 0 \\ -\omega x & -\omega^2 xy & \omega^2 x^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T^{\alpha\beta}_{tot} = T^{\alpha\beta}_{cw} + T^{\alpha\beta}_{ccw} = \boxed{2\rho_0 \gamma^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega^2 y^2 & -\omega^2 xy & 0 \\ 0 & -\omega^2 xy & \omega^2 x^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

4. Use the identity $\partial_\nu T^{\mu\nu} = 0$ to prove the following results for a bounded system (i.e. a system for which $T^{\mu\nu} = 0$ beyond some bounded region of space)

a. Show the expression for conservation of energy + momentum:

$$\partial_t \int T^{0\alpha} d^3x = 0$$

· Start w/ the fact that $\partial_\mu T^{\nu\mu} = 0$

· Symmetry of $T^{\nu\mu} \longleftrightarrow T^{\mu\nu}$

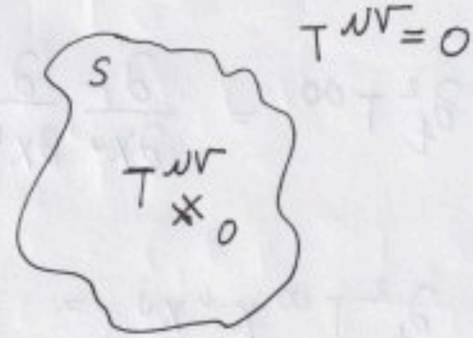
$$\Rightarrow \partial_\mu T^{\mu\nu} = 0$$

$$\Rightarrow \partial_t T^{0\nu} + \frac{\partial T^{j\nu}}{\partial x^j} = 0 \qquad \text{relabel } \nu \to \alpha$$

$$\Rightarrow \partial_t T^{0\alpha} + \frac{\partial T^{j\alpha}}{\partial x^j} = 0$$

Integrate over the V^3 separating $T^{\nu\mu} = 0$ + $T^{\nu\mu} \neq 0$

$$\Rightarrow \int_{V^3} \partial_t T^{0\alpha} \, d^3x = -\int_{V^3} \frac{\partial T^{j\alpha}}{\partial x^j} \, d^3x$$



$$\Rightarrow \frac{\partial}{\partial t} \int_{V^3} T^{0\alpha} \, d^3x = -\oint_S T^{\beta\alpha} \, d\Sigma_\beta$$

Via analogue of Gauss' Theorem

$$\Rightarrow \boxed{\frac{\partial}{\partial t} \int_{V^3} T^{0\alpha} \, d^3x = 0}$$

since along boundary of S, $T^{\nu\mu} = 0$ uniformly

b · Show that $\partial_t^2 \int T^{00} x^i x^j \, d^3x = 2 \int T^{ij} \, d^3x$

· This is a version of the virial theorem.

· Start with $\partial_t T^{0\alpha} + \dfrac{\partial T^{j\alpha}}{\partial x^j} = 0$ ☆

· Take a 2nd ∂_t on both sides:

$$\partial_t^2 T^{0\alpha} = \frac{\partial}{\partial t}\frac{\partial}{\partial x^j} T^{j\alpha}$$ · Now commute partials

$$= \frac{\partial}{\partial x^j}\partial_t T^{j\alpha}$$ · Now choose $\alpha = 0$

$$\partial_t^2 T^{00} = \frac{\partial}{\partial x^j}\partial_t T^{j0} = \frac{\partial}{\partial x^j}\partial_t T^{0j}$$ symmetry

· Now apply ☆ again

$$\partial_t^2 T^{00} = \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i} T^{ij}$$ · Now multiply by x^i, x^j

$$\partial_t^2 T^{00} x^i x^j = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} T^{ij} x^i x^j$$

$$= \left(\frac{\partial}{\partial x^i}\right)\left(x^i T^{ij} \overbrace{\frac{\partial x^i}{\partial x^j}}^{\delta^i_j} + x^i T^{ij} \overbrace{\frac{\partial x^j}{\partial x^j}}^{1} + \overbrace{\frac{\partial T^{ij}}{\partial x^j}}^{0} x^i x^j\right)$$

$$= \frac{\partial}{\partial x^i}\left(T^{ij}x^i + T^{ij}x^i + 0\right)$$

$$= 2T^{ij}$$ · Now integrate over V^3

$$\boxed{\partial_t^2 \int T^{00} x^i x^j \, d^3x = 2\int T^{ij} d^3x}$$ ✓ ☺

$\boxed{c}$. Show that $\partial_t^2 \int T^{00} (x^i x_i)^2 \, d^3x$

$$= 4\int T^i_i \, x^j x_j \, d^3x + 8 \int T^{ij} x_i x_j \, d^3x$$

- (No pithy wisdom for this equation / no good interpretation ... 😐)
- From past part of problem, start with our given as:

$\partial_t^2 T^{00} = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} T^{ij} \equiv \partial_i \partial_j T^{ij}$ Now multiply by $x^i x_i x^j x_j$ ...

$$\partial_t^2 T^{00} x^i x_i x^j x_j = \partial_i \partial_j T^{ij} x^i x_i x^j x_j$$

$$\partial_t^2 T^{00} (x^i x_i)^2 = \partial_i \Big(T^{ij} \delta^i_j x_i x^j x_j + T^{ij} x^i \delta_{ij} x^j x_j + T^{ij} x^i x_i \delta^j_j x_j + T^{ij} x^i x_i x^j \delta_{jj} \Big)$$

$$= \partial_i \Big(T^{ij} x^i x_i x_j + T^{ij} x^i x_i x_j + T^{ij} x^i x_i x_j + T^i_j x^i x_i x^j \Big)$$

$$= \partial_i \Big(3 T^{ij} x^i x_i x_j + T^i_j x^i x_i x^j \Big)$$

$=$

$$= 3T^{ij}\left(\delta^i_i x_i x_j + x^i \delta_{ii} x_j + x^i x_i \delta_{ij}\right)$$

$$+ T^i_j\left(\delta^i_i x_i x^j + x^i \delta_{ii} x^j + x^i x_i \delta^j_i\right)$$

$$= 3T^{ij}\left(\boxed{x_i x_j + x_i x_j} + x^i x_i \delta_{ij}\right)$$

$$+ T^i_j\left(\boxed{x_i x^j + x^i \delta_{ii} x^j} + x^i x_i \delta^j_i\right)$$

$$\bigcirc = 3T^i_i x^i x_i + T^i_i x^i x_i$$

$$= 4T^i_i x^j x_j$$

$$\boxed{} = 6T^{ij} x_i x_j + \underbrace{T^i_j x_i x^j}_{\substack{\downarrow \\ \text{same trick} \\ \downarrow \\ T^{ij} x_i x_j}} + \underbrace{T^i_j x^i \delta_{ii} x^j}_{\substack{\downarrow \\ T^i_j x_i x^j \\ \text{(use the fact that } v^\alpha v_\alpha = v^\alpha v_\alpha \text{ to} \\ \text{flip } j \text{ up + } j \text{ down)} \\ \downarrow \\ T^{ij} x_i x_j}}$$

$$= 8T^{ij} x_i x_j$$

· So overall, we get that:

$$\partial_t^2 T^{00} (x^i x_i)^2 = 4T^i_i x^j x_j + 8T^{ij} x_i x_j$$

which we can integrate over to find the desired result ✓

$\boxed{5}$. The vector potential $\vec{A} \doteq (A^0, \underset{\sim}{A})$ generates the electromagnetic field tensor via

$$F_{\nu\gamma} = \partial_\nu A_\gamma - \partial_\gamma A_\nu$$

$\boxed{a}$. From the above info, derive that

$$\underset{\sim}{B} = \overline{\nabla} \times \underset{\sim}{A} \quad \text{and}$$

$$\underset{\sim}{E} = -\frac{\partial}{\partial t}\underset{\sim}{A} - \nabla A^0$$

where "Nabla" "∇" is the Euclidean gradient:

· From the lecture notes:

$$F^{\nu\gamma} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix}$$

$$\Rightarrow \quad B_x \hat{x} = F^{23}\hat{x} = (\partial_y A_z - \partial_z A_y)\hat{x}$$

$$B_y \hat{y} = F^{31}\hat{y} = (\partial_z A_x - \partial_x A_z)\hat{y}$$

$$B_z \hat{z} = F^{12}\hat{z} = (\partial_x A_y - \partial_y A_z)\hat{z}$$

· compare this to:

$$\underset{\sim}{\nabla} \times \underset{\sim}{A} \equiv \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix}_{det}$$

$$= (\partial_y A_z - \partial_z A_y)\hat{x} + (\partial_z A_x - \partial_x A_z)\hat{y} + (\partial_x A_y - \partial_y A_x)\hat{z}$$

$$\Rightarrow \boxed{\underset{\sim}{B} = \underset{\sim}{\nabla} \times \underset{\sim}{A}} \text{ indeed } \checkmark$$

. Now to prove the $\underset{\sim}{E}$ equation:

· Looking at $F^{\mu\nu}$ we see that $F^{0i} = E_i$

-or- in other words - the first row gives us the spatial components of $\underset{\sim}{E}$

· <u>Now</u>

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$$
$$= \eta^{\mu\alpha} \partial_{\alpha} A^{\nu} - \eta^{\nu\beta} \partial_{\beta} A^{\mu}$$

$$\Rightarrow F^{0i} = \eta^{0\alpha} \partial_{\alpha} A^{i} - \eta^{i\beta} \partial_{\beta} A^{0}$$

· Writing this out in matrix notation and remmembering $\eta^{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$

$$E_i = F^{0i} = [-1, 0, 0, 0]\begin{bmatrix}\partial_t A^i \\ \partial_x A^i \\ \partial_y A^i \\ \partial_z A^i\end{bmatrix} - \begin{bmatrix}0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\partial_t A^0 \\ \partial_x A^0 \\ \partial_y A^0 \\ \partial_z A^0\end{bmatrix}$$

$$\Rightarrow E_i = -\partial_t A^i - (\partial_x + \partial_y + \partial_z) A^0$$

$$\Rightarrow \boxed{\underset{\sim}{E} = -\partial_t \underset{\sim}{A} - \nabla A^0}$$

actually choose one of these rows depending on which E_i you want...

this should actually just be ∂_i not $\partial_x + \partial_y + \partial_z$ ...

[b] · Show that Maxwell's equations only hold if:

$$\partial_\nu \partial^\nu A^\alpha - \partial^\alpha \partial_\nu A^\nu = -4\pi J^\alpha$$

· From the notes we know that:

$$\begin{aligned}\partial_\gamma F^{\nu\gamma} &= 4\pi J^\nu \\ &= \partial_\gamma(\partial^\nu A^\gamma - \partial^\gamma A^\nu) \\ &= -\partial_\gamma \partial^\gamma A^\nu + \partial_\gamma \partial^\nu A^\gamma\end{aligned}$$

· Let $\gamma \to \nu$, $\nu \to \alpha$

$$\Rightarrow \boxed{-4\pi J^\alpha = \partial_\nu \partial^\nu A^\alpha - \partial_\nu \partial^\alpha A^\nu}$$

c

· Let $A_\nu^{new} = A_\nu^{old} + \partial_\nu \phi$

$$\Rightarrow F^{\nu\tau}_{new} \to \partial_\nu A_\tau^{new} - \partial_\tau A_\nu^{new}$$

$$= \partial_\nu (A_\tau^{old} + \partial_\tau \phi) - \partial_\tau (A_\nu^{old} + \partial_\nu \phi)$$

$$= (\partial_\nu A_\tau^{old} - \partial_\tau A_\nu^{old}) + \underbrace{(\partial_\nu \partial_\tau \phi - \partial_\tau \partial_\nu \phi)}_{= \emptyset}$$

$$= \boxed{F^{\nu\tau}_{old}}$$

d

· If [redacted] we choose ϕ such that

$\boxed{\partial_\nu \phi = -A_\nu^{old}}$ then

$$A_\nu^{new} = A_\nu^{old} + \partial_\nu \phi$$

$$= A_\nu^{old} - A_\nu^{old}$$

$= \emptyset$ and yet we still have $F^{\nu\tau}$ unchanged by this gauge

· In this gauge;

$$[redacted] \quad \partial_\nu \partial^\nu A^\alpha - \partial^\alpha \overbrace{\partial_\nu A^\nu}^{0} = -4\pi J^\alpha$$

$$\to \partial_\nu \partial^\nu A^\alpha = -4\pi J^\alpha$$

$$\to \boxed{J^\alpha = \frac{-\Box A^\alpha}{4\pi} \quad \text{where} \quad \Box = \partial_\nu \partial^\nu}$$

$\boxed{6}$. An astronaut is accelerating in the x-direction with 4-acceleration $\vec{a}\cdot\vec{a} = g^2$. The astronaut defines his coords as $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$. We can ignore $\bar{y}$ + $\bar{z}$ since there is no motion in these directions. $\bar{t}$ is the astronaut's own proper time. At $\bar{t} = 0$, the astronaut's coords momentarily line up with Coordinate Stationary observer's (CSOs) who define their coords as (t, x, y, z). I.e. at $\bar{t} = 0$, $t = \bar{t}$. We define a function "A" that converts between coordinate $\bar{t}$ and "t" as:

$$A \equiv d\bar{t}/d\tau \leftarrow \text{coord. stationary observer's proper time}$$

The astronaut requires that the worldlines of CSOs must be orthogonal to the hyper-surfaces $\bar{t}$ = constant + that for each $\bar{t}$ there exists an inertial frame momentarily at rest w.r.t. the astronaut in which all events with $\bar{t}$ = constant are simultaneous.

$\boxed{a}$. What is the 4-velocity of the astronaut as a function of $\bar{t}$ in the initial IRF that uses coords (t, x, y, z)?

· We know that:

$$\vec{u}\cdot\vec{u} = -1 \rightarrow -(U^0)^2 + (U^1)^2 = -1$$

$$\vec{u}\cdot\vec{a} = 0 \rightarrow -a^0 U^0 + U^1 a^1 = 0$$

$$\vec{a}\cdot\vec{a} = g^2 \rightarrow -(a^0)^2 + (a^1)^2 = g^2$$

· Also, $a^0 \equiv \frac{dU^0}{d\bar{t}}$ and $a^1 \equiv \frac{dU^1}{d\bar{t}}$

· Combining all of these we get a set of diff. eq. relations:

$$-(U^0)^2 + (U^1)^2 = -1 \quad \text{(i)}$$

$$-\frac{dU^0}{d\bar{t}} U^0 + U^1 \frac{dU^1}{d\bar{t}} = 0 \quad \text{(ii)}$$

$$-\left(\frac{dU^0}{d\bar{t}}\right)^2 + \left(\frac{dU^1}{d\bar{t}}\right)^2 = g^2 \quad \text{(iii)}$$

· Choose $U^0 = \cosh(A\bar{t})$, $U^1 = \sinh(A\bar{t})$

for (i) $\sinh^2(A\bar{t}) - \cosh^2(A\bar{t}) = -1$ ✓

for (ii) $-A\sinh(\cancel{A\bar{t}})\cosh(A\bar{t}) + A\sinh(\cancel{A\bar{t}})\cosh(A\bar{t}) = 0$ ✓

for (iii) $-A^2\sinh^2(A\bar{t}) + A^2\cosh^2(A\bar{t}) = A^2 = g^2$

$$\Longrightarrow A = g \rightsquigarrow$$

· So we get that:

$$\boxed{\vec{u} = (\cosh(g\bar{t}), \sinh(A\bar{t}), 0, 0)}$$

4-velocity of astronaut in initial IRF where $t = \bar{t}$ momentarily

Rather than continuing w/ part $\boxed{b}$ next, I will derive the relations for part $\boxed{d}$ and then parts $\boxed{b}$, $\boxed{c}$, and $\boxed{e}$ follow naturally:

$\boxed{d}$ · Find explicit transformations for $x(\bar{t}, \bar{x})$ and $t(\bar{t}, \bar{x})$. These are known as Kottler-Møller coordinates. We will follow a derivation that Prof. Scott Hughes did in his special relativity notes available online:

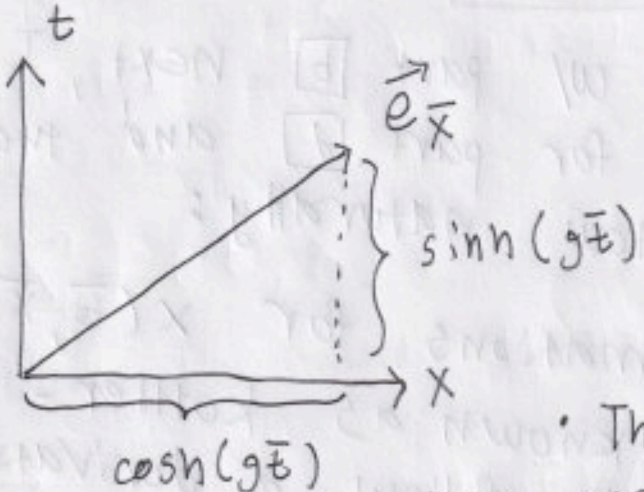
· We define $\vec{e}_{\bar{t}}$ to be the unit-time vector in the accelerating astronaut's frame which lies parallel to the 4-velocity we just found above:

$$\vec{e}_{\bar{t}} \equiv \cosh(g\bar{t})\,\vec{e}_t + \sinh(g\bar{t})\,\vec{e}_x$$

· We then define $\vec{e}_{\bar{x}}$ to be the unit-spatial vector orthogonal to $\vec{e}_{\bar{t}}$:

$$\vec{e}_{\bar{x}} \equiv \sinh(g\bar{t})\,\vec{e}_t + \cosh(g\bar{t})\,\vec{e}_x$$

· Now with the definition of $\vec{e}_{\bar{x}}$, consider the surface defined by $\bar{t}$ = constant. This must be parallel to $\bar{x}$ axis:



i.e. this surface of constant $\bar{t}$ must have the slope in the x-t plane of:

$$m = \tanh(g\bar{t})$$

· Therefore, the equation for the surface of constant $\bar{t}$ in the x-t plane must be:

$$y = mx \rightarrow t = x\tanh(g\bar{t}) \quad ⍟$$

· Now what about a surface of constant $\bar{x}$?

This must lie parallel to the $\vec{e}_{\bar{t}}$ vector:

$$\frac{dt}{dx} = m = \coth(g\bar{t})$$ [redacted]

$$\Rightarrow \frac{\partial x}{\partial t} = \tanh(g\bar{t}) \quad ⊛$$

⍟ + ⊛ combine to:

$$\frac{\partial x}{\partial t} = \frac{t}{x} \quad \rightsquigarrow$$

$$\int_{\bar{x}}^{x} x\,dx = \int_{0=\bar{t}}^{t} t\,dt$$ integrate as the next step

$$\Rightarrow \quad x^2 - \bar{x}^2 = t^2$$

· This kind of equation represent hyperbola where $x = \bar{x}\cosh(\alpha)$ and $t = \bar{x}\sinh(\alpha)$ for some α. Plug these back into (A) to find:

$$x(\bar{x},\bar{t}) = \bar{x}\cosh(g\bar{t})$$
$$t(\bar{x},\bar{t}) = \bar{x}\sinh(g\bar{t})$$

· But wait! These do not have the proper behavior that $x = 0$ @ $t = \bar{t} = 0$. To do this we need to shift $\bar{x} \rightarrow \bar{x} + \frac{1}{g}$ and

$$x(\bar{x},\bar{t}) \rightarrow x(\bar{x},\bar{t}) - \frac{1}{g}$$

· So the final solution is:

$$\boxed{\begin{aligned} x &= \left(\bar{x} + \tfrac{1}{g}\right)\cosh(g\bar{t}) - \tfrac{1}{g} \\ t &= \left(\bar{x} + \tfrac{1}{g}\right)\sinh(g\bar{t}) \end{aligned}}$$

"Kottler-Møller"

6 . Imagine that each coordinate-stationary observer carries a clock. What is the 4-velocity of each clock in the initial inertial frame:

- To do this, we will write the [redacted] 4-vector of a given CSO in its own coords:

$$\vec{X}_{cso} = (t, x)$$

- Now use our transformations

$$\vec{X}_{cso} = \left((\bar{x} + \tfrac{1}{g})\sinh(g\bar{t}), (\bar{x} + \tfrac{1}{g})\cosh(g\bar{t}) - \tfrac{1}{g} \right)$$

- Now take $\frac{d}{dt} \leftarrow$ CSO proper time

$$\frac{d}{dt} = \frac{dt}{d\bar{t}}\frac{d}{d\bar{t}} = A\frac{d}{d\bar{t}}$$

$$\Rightarrow \vec{u}_{cso} = A\left((1+g\bar{x})\cosh(g\bar{t}), (1+g\bar{x})\sinh(g\bar{t}) \right)$$

However, the $\vec{u}_{cso} \cdot \vec{u}_{cso}$ must $= -1$

$$\Rightarrow A^2(1+g\bar{x})^2 \underbrace{(\sinh^2() - \cosh^2())}_{-1} = -1$$

$$\Rightarrow A = \frac{1}{1+g\bar{x}}$$

$$\Rightarrow \boxed{\vec{u}_{cso} = (\cosh(g\bar{t}), \sinh(g\bar{t}), 0, 0)}$$

i.e. the 4-velocity of a CSO's clock at $t=\bar{t}=0$ in the coord-representation of the CSO is actually equal to the 4-velocity of the accelerating astronaut at $t=\bar{t}=0$ in the coord-representation of the astronaut...

$\boxed{c}$. Show that $A(\bar{t},\bar{x},\bar{y},\bar{z})$ actually is just a function of $\bar{x}$. There may be an intellectual way to do this thoughtfully, but we actually just showed in the previous part that:

$$\boxed{A(\bar{x}) = \frac{1}{1+g\bar{x}}}$$

$\boxed{e}$. Find ds^2 in both representations:

$$ds^2 = dx^2 - dt^2$$

Also $dx = \left(\bar{x} + \frac{1}{g}\right)\sinh(g\bar{t})\, g\, d\bar{t} + d\bar{x}\cosh(g\bar{t})$

$$dt = (g\bar{x}+1)\cosh(g\bar{t})\, d\bar{t} + d\bar{x}\sinh(g\bar{t})$$

$$\Rightarrow ds^2 = -(g\bar{x}+1)^2\cosh^2()\, d\bar{t}^2 - \sinh^2()\, d\bar{x}^2$$
$$- 2(g\bar{x}+1)\cancel{\sinh()\cosh()}\, d\bar{x}\, d\bar{t}$$
$$+ (\bar{x}g+1)^2\sinh^2()\, d\bar{t}^2 + \cosh^2()\, d\bar{x}^2$$
$$+ 2(g\bar{x}+1)\cancel{\sinh()\cosh()}\, d\bar{x}\, d\bar{t}$$

Implying:

$$\boxed{\begin{aligned} ds^2 &= dx^2 - dt^2 \\ &= d\bar{x}^2 - (1+g\bar{x})^2 d\bar{t}^2 \end{aligned}}$$